

An improvement on the trivial lower bound for the depth of a centerline

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1 Introduction

Let α be a k -flat and μ a probability measure in \mathbb{R}^d ($0 \leq k < d$). Define the *depth* of α as follows:

$$\text{depth}_\mu(\alpha) = \inf\{\mu(H) : H \text{ is a closed half-space, } \alpha \subset \partial H\}.$$

To distinguish with other notions of depth, the above defined depth is sometimes called *half-space depth* or *Tukey depth*. We will write simply $\text{depth}(\alpha)$ if the measure is clear from the context.

Throughout the paper we will deal only with probability measures. Thus we will omit the word “probability”, however, keeping it in mind.

Bukh, Matoušek and Nivasch [2] proposed the following conjecture:

Conjecture 1.1. *Let (d, k) be a pair of integers with $0 \leq k < d$. Then for every measure μ in \mathbb{R}^d there exists a k -flat α in \mathbb{R}^d (a centerflat) such that*

$$\text{depth}(\alpha) \geq \frac{k+1}{k+d+1}. \quad (1)$$

The conjecture is true for $k = 0$ (Radó’s centerpoint theorem, 1946, see [6]), $k = d - 1$ (trivial), and $k = d - 2$ (due to Bukh, Matoušek and Nivasch [2]).

A result by Klartag [5] implies that, if $d - k$ is fixed, then for every $\varepsilon > 0$, with d sufficiently large depending on ε , and for every measure μ in \mathbb{R}^d there exists a k -flat α in \mathbb{R}^d such that

$$\text{depth}(\alpha) > \frac{1}{2} - \varepsilon.$$

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One can see that for $k = 0$ and $k = d - 1$ the constant $\frac{k+1}{k+d+1}$ in (1) cannot be increased. This is also the case for $k = 1$, as shown by Buch and Nivasch [3].

Radó's centerpoint theorem implies that for every d , k and μ as above one can find a k -flat α such that

$$\text{depth}(\alpha) \geq \frac{1}{d - k + 1}. \quad (2)$$

(In fact, such a k -flat exists in any k -dimensional direction.) The bound of (2) will be called the *trivial bound*.

It may be also convenient to think about a depth of a flat in terms of projections. If μ is a measure in \mathbb{R}^d , and α is a k -flat, we will write π_β for the projection from \mathbb{R}^d onto \mathbb{R}^{d-k} along α . Let μ_α be a projection of μ along α , i.e., a measure in \mathbb{R}^{d-k} such that for every Borel set $X \subseteq \mathbb{R}^{d-k}$ one has

$$\mu_\alpha(X) = \mu(\pi_\alpha^{-1}(X)).$$

Then for the point $o = \pi_\alpha(\alpha)$ one has the identity

$$\text{depth}_{\mu_\alpha}(o) = \text{depth}_\mu(\alpha).$$

In this paper we prove that for $k = 1$ the trivial bound (2) is not optimal, except for the case $d = 2$. Namely, we have the following result:

Theorem 1.2. *For every $d \geq 3$ and for every probability measure μ in \mathbb{R}^d there exists a (1-dimensional) line ℓ with*

$$\text{depth}(\ell) \geq \frac{1}{d} + \frac{1}{3d^3}. \quad (3)$$

We emphasize that there is still a huge gap between the inequality (3) we were able to prove, and the conjectured inequality (1).

Theorem 1.2 also implies that the trivial bound (2) is optimal *only* for the cases $k = 0$ and $k = d - 1$, as stated in the following corollary.

Corollary 1.3. *For every $d \geq 3$, every k such that $1 \leq k \leq d - 2$ and every probabilistic measure μ in \mathbb{R}^d there exists a k -dimensional flat α with*

$$\text{depth}(\alpha) \geq \frac{1}{d - k + 1} + \frac{1}{3(d - k + 1)^3}.$$

Proof (reduction to Theorem 1.2. Choose an arbitrary $(k-1)$ -dimensional flat β . After projecting along β onto \mathbb{R}^{d-k+1} we can apply Theorem 1.2. Namely, we conclude that there is a line $\ell \subset \mathbb{R}^{d-k+1}$ such that

$$\text{depth}_{\mu_\beta}(\ell) \geq \frac{1}{d - k + 1} + \frac{1}{3(d - k + 1)^3}.$$

To finish the proof it is enough to put

$$\alpha = \pi_\beta^{-1}(\ell).$$

□

2 Outline of the proof

First, it will be convenient for us to prove Theorem 1.2 for the $(d+1)$ -dimensional space rather than for the d -dimensional. Next, we aim for a proof by contradiction. Therefore we assume that for every one-dimensional direction ℓ no point of the $(d$ -dimensional) plane $\pi_\ell(\mathbb{R}^{d+1})$ has depth

$$a_0 = a_0(d+1) = \frac{1}{d+1} + \frac{1}{3(d+1)^3}$$

or greater.

Topological setup. We write $\mathbb{R}P^d$ for the space of all one-dimensional directions \mathbb{R}^{d+1} as this is indeed the real projective space.

Let $\xi = (E, \mathbb{R}P^d, p)$ be the quotient tautological bundle over $\mathbb{R}P^d$. That is, for every $\ell \in \mathbb{R}P^d$ the fiber $p^{-1}(\ell)$ is $\mathbb{R}^d = \pi_\ell(\mathbb{R}^{d+1})$. Hence we can assume that $p^{-1}(\ell)$ is equipped with the measure μ_ℓ .

We prove that there is no $(d+1)$ -fold space cover $E' \subset E$ with a certain property (namely, that the origin of each fiber $p^{-1}(\ell)$ is contained in the convex hull of the $(d+1)$ -point set $p^{-1}(\ell) \cap E'$; see Section 8). Hence, if any assumption allows us to construct such E' , then the assumption is false.

Nice measures. We will call a measure *nice*, if it has an everywhere positive density, and the density function decreases at least as an exponent of radius. We supply the space of nice measures with a metric — the L^1 distance between density functions. If a measure μ is nice, then every its projection μ_α is nice, and also μ_α is continuous with α .

In the proof of Theorem 1.2 we will assume that μ is nice. Then the case of an arbitrary μ will follow from a standard approximation argument.

Centerpoints of measures μ_ℓ . We show that each μ_ℓ has a unique centerpoint, i.e., a point at the largest depth. In the proof we use that all points of $p^{-1}(\ell)$ have small depth with respect to μ_ℓ . We put the origin of each fiber at its centerpoint (it is possible, because we show the continuity of the centerpoint). The necessary lemmas concerning the centerpoint of a measure are proved in Section 3.

Generating $(d+1)$ -tuples of half-spaces. Let $\mathbb{R}^d = p^{-1}(\ell)$ be a fiber and $\mathbf{0}$ its origin. If a $(d+1)$ -tuple of closed half-spaces

$$H_1, H_2, \dots, H_{d+1} \subseteq \mathbb{R}^d$$

satisfies

$$\bigcap_{i=1}^{d+1} H_i = \mathbf{0},$$

then we call it *generating*.

For such a $(d+1)$ -tuple of half-spaces we can consider the *corresponding* $(d+1)$ -tuple of simplicial cones

$$B_1, B_2, \dots, B_{d+1},$$

namely,

$$B_i = \bigcap_{\substack{1 \leq j \leq d+1 \\ j \neq i}} H_j. \quad (4)$$

Clearly, different $(d+1)$ -tuples of half-spaces generate different $(d+1)$ -tuples of simplicial cones.

A generating $(d+1)$ -tuple of half-spaces is said to have *weight* a , if

$$\min_i \mu_\ell(H_i) = 1 - a.$$

Generating $(d+1)$ -tuples of small weight. A generating $(d+1)$ -tuple of half-spaces is, informally, a $(d+1)$ -tuple with weight equal to $\frac{1}{d+1}$ or only slightly greater. In Section 4 we pay special attention to generating $(d+1)$ -tuples of small weight. Indeed, as we prove in Section 3, such $(d+1)$ -tuples exist in every fiber $p^{-1}(\ell)$.

We show that all generating $(d+1)$ -tuples of small weight are in a certain sense similar to each other. For instance, given an order for one such $(d+1)$ -tuple, a natural order for all other such $(d+1)$ -tuples can be introduced. Moreover, a natural order is spanned for generating $(d+1)$ -tuples of small weight in $p^{-1}(\ell')$ if ℓ' is close enough to ℓ .

Constructing the space cover. In Section 5, given a measure μ and a convex d -dimensional convex cone C in \mathbb{R}^d , we define the *central vector* of C . This definition is used to obtain a space cover $E' \subset E$. The argument goes roughly as follows. Given a fiber $p^{-1}(\ell)$ consider all generating $(d+1)$ -tuples of small weight, and take the i -th cone $B_i(\mathcal{H})$ corresponding to each such $(d+1)$ -tuple \mathcal{H} . Then average the central vector of $B_i(\mathcal{H})$ over all \mathcal{H} . This would give the i -th point of the set $E' \cap p^{-1}(\ell)$. All the details are provided in Section 7.

3 The centerpoint of a measure

If μ is a nice measure in \mathbb{R}^d , then $\text{depth}(x)$, considered as a function of a point $x \in \mathbb{R}^d$, is continuous. Since the limit of $\text{depth}(x)$ is zero at infinity, then $\text{depth}(x)$ attains its maximal value.

Given a nice measure μ , call a point o a *centerpoint* if

$$\text{depth}(x) \leq \text{depth}(o) \quad \text{for all } x \in \mathbb{R}^d.$$

We are going to prove the uniqueness of the centerpoint, provided that its depth is small. The idea of such statement is certainly not new, see, for instance, [1].

If $n \in \mathbb{R}^d$ is a unit vector, denote by $H(n)$ the half-space such that the origin $\mathbf{0}$ belongs to $\partial H(n)$ and n is the outer normal to $\partial H(n)$, i.e., n is orthogonal to $\partial H(n)$ and is directed outwards the half-space $H(n)$.

Lemma 3.1. *Let $0 \leq \varepsilon < \frac{1}{3(d+1)^3}$, $a = \frac{1}{d+1} + \varepsilon$. If $\mathbf{0}$ is a centerpoint of μ and $\text{depth}(\mathbf{0}) = a$, then*

1. *There exists a generating $(d+1)$ -tuple of half-spaces H_1, H_2, \dots, H_{d+1} of weight a .*
2. *$\mathbf{0}$ is the unique centerpoint of μ .*

Proof. Define \mathcal{N} to be the set of all unit vectors $n \in \mathbb{R}^d$ such that $\mu(H(n)) = a$. Clearly, the set \mathcal{N} is compact.

Assume that $\mathbf{0} \notin \text{conv } \mathcal{N}$. Then $\text{conv } \mathcal{N}$ can be split from $\mathbf{0}$ by a plane. Or, equivalently, there exists a unit vector v such that

$$\inf_{n \in \mathcal{N}} \langle n, v \rangle > 0.$$

This is impossible, since for small enough $\delta > 0$ we have $\text{depth}(\delta \cdot v) > a$. Indeed, if we translate μ by a vector $-\delta \cdot v$, then $\text{depth}(\mathbf{0})$ increases. The reason is that translation of μ by $-\delta \cdot v$ increases the measure of each $H(n)$ for all n in some open neighborhood of \mathcal{N} and does not sufficiently decrease the measure of all other half-spaces with $\mathbf{0}$ in the boundary. The contradiction shows that $\mathbf{0} \in \text{conv } \mathcal{N}$.

The Carathéodory principle implies that

$$\mathbf{0} \in \text{conv}\{n_1, n_2, \dots, n_k\} \quad (n_i \in \mathcal{N}, 2 \leq k \leq d+1).$$

We claim that $k = d+1$. Indeed, by the choice of n_i we have

$$\mu \left(\bigcap_{i=1}^k H(-n_i) \right) = 0,$$

because the intersection is a $(d+1-k)$ -dimensional affine plane through o . But this is not possible for $k \leq d$, because

$$\mu(H(-n_i)) = 1 - a > 1 - \frac{1}{d}.$$

Consequently, the $(d+1)$ -tuple of half-spaces

$$H(-n_1), H(-n_2), \dots, H(-n_{d+1})$$

is generating and has weight a . Assertion 1 is proved.

For assertion 2, let $o \neq \mathbf{0}$. For some i we have $o \in \text{int } H(n_i)$. Then for the half-space H such that

$$o \in \partial H, \quad \partial H \perp n_i, \quad \text{and} \quad H \subset H(n_i)$$

we have $\mu(H) < \mu(H(n_i)) = a$. Hence $\text{depth}(o) < a$.

□

Now let the measure μ change. Recall that we consider the space of nice probability measures in \mathbb{R}^d with L^1 -distance between density functions as the metric. Notice that for every measurable set X one has

$$|\mu(X) - \mu'(X)| \leq \|f_\mu - f_{\mu'}\|_{L^1}.$$

Here f_μ and $f_{\mu'}$ denote the density functions of the respective measures.

Lemma 3.2. *Let $o(\mu)$ denote a centerpoint of μ . Assume that μ_0 is a measure such that*

$$\text{depth}_{\mu_0}(o(\mu_0)) < \frac{1}{d+1} + \frac{1}{3(d+1)^3}.$$

Then

1. $o(\mu)$ is unique for all μ in some neighborhood of μ_0 .
2. $\text{depth}_\mu(o(\mu))$ is continuous with μ at the point $\mu = \mu_0$.
3. $o(\mu)$ is continuous with μ at the point $\mu = \mu_0$.

Proof. Without loss of generality, let $o(\mu_0) = \mathbf{0}$. Let also $\text{depth}_{\mu_0}(o) = a$.

If

$$\|f_\mu - f_{\mu_0}\|_{L^1} < \frac{1}{d+1} + \frac{1}{3(d+1)^3} - a,$$

then, for the measure μ , every point in \mathbb{R}^d has depth less than $\frac{1}{d+1} + \frac{1}{3(d+1)^3}$. Application of Lemma 3.1 yields assertion 1.

Further,

$$\begin{aligned}
a - \|f_\mu - f_{\mu_0}\|_{L^1} &= \text{depth}_{\mu_0}(\mathbf{0}) - \|f_\mu - f_{\mu_0}\|_{L^1} \leq \text{depth}_\mu(\mathbf{0}) \\
&\leq \text{depth}_\mu(o(\mu)) \leq \\
\text{depth}_{\mu_0}(o(\mu)) + \|f_\mu - f_{\mu_0}\|_{L^1} &\leq \text{depth}_{\mu_0}(\mathbf{0}) + \|f_\mu - f_{\mu_0}\|_{L^1} = a + \|f_\mu - f_{\mu_0}\|_{L^1}. \quad (5)
\end{aligned}$$

Hence assertion 2 follows.

According to Lemma 3.1, there is a generating $(d+1)$ -tuple of half-spaces

$$H_1, H_2, \dots, H_{d+1}$$

of weight a , and thus $\mu_0(H_i) = 1 - a$.

For any open ball $U \subset \mathbb{R}^d$ such that $\mathbf{0} \in U$ there exists a set of half-spaces

$$H'_1, H'_2, \dots, H'_{d+1}$$

such that

$$\partial H_i \parallel \partial H'_i, \quad H_i \subset \text{int } H'_i, \quad \text{and} \quad S = \bigcup_{i=1}^{d+1} H'_i \subset U.$$

By construction, S is a d -simplex, and $\mathbf{0} \in \text{int } S$.

Denote

$$\delta = \min_{1 \leq i \leq d+1} \mu_0(H'_i) - (1 - a).$$

Since μ_0 is nice, $\delta > 0$.

Let

$$\|f_\mu - f_{\mu_0}\|_{L^1} < \min\left(\frac{\delta}{2}, \frac{1}{d+1} + \frac{1}{3(d+1)^3} - a\right).$$

Suppose that $o = o(\mu) \notin S$. According to (5), $\text{depth}_\mu(o(\mu)) > a - \frac{\delta}{2}$. On the other hand, for some i we have $o \in \mathbb{R}^d \setminus H'_i$. Therefore

$$\text{depth}_\mu(o) \leq \mu(\mathbb{R}^d \setminus H'_i) \leq \mu_0(\mathbb{R}^d \setminus H'_i) + \|f_\mu - f_{\mu_0}\|_{L^1} < a - \delta + \frac{\delta}{2} = a - \frac{\delta}{2},$$

a contradiction. Thus $o(\mu) \in S \subset U$. But U is arbitrary, so $o(\mu)$ is continuous. \square

4 Generating $(d+1)$ -tuples of half-spaces

In this section we prove several auxiliary facts concerning generating $(d+1)$ -tuples of cones.

Recall that a generating $(d+1)$ -tuple of half-spaces generates a $(d+1)$ -tuple of simplicial cones according to formula (4).

Lemma 4.1. *Let*

$$H_1, H_2, \dots, H_{d+1}$$

be a generating $(d + 1)$ -tuple of half-spaces and

$$B_1, B_2, \dots, B_{d+1}$$

the corresponding $(d + 1)$ -tuple of simplicial cones. Let for each i a point $b_i \in \text{int } B_i$ be chosen. Then $\text{conv}\{b_1, b_2, \dots, b_{d+1}\}$ is a non-degenerate d -simplex, and

$$\mathbf{0} \in \text{int conv}\{b_1, b_2, \dots, b_{d+1}\}.$$

Proof. For each plane ∂H_i denote by n_i the unit normal vector directed outwards H_i . We have

$$\mathbf{0} \in \text{int conv}\{n_1, n_2, \dots, n_{d+1}\}, \quad (6)$$

otherwise

$$\bigcap_{i=1}^{d+1} H_i \neq \{\mathbf{0}\}.$$

By construction, for every $i \neq j$ we have

$$\langle b_i, n_i \rangle > 0, \quad \langle b_i, n_j \rangle < 0.$$

Now we argue by contradiction. Assuming that the statement of lemma is false, there is a plane α that separates $\mathbf{0}$ from every b_i . (The separation need not be strict.) If n is the normal vector to α pointing towards the open half-space without any b_i , then

$$\langle b_i, n \rangle \geq 0 \quad \text{for all } i.$$

Due to (6), we may assume without loss of generality that

$$n = \lambda_1 n_1 + \lambda_2 n_2 + \dots + \lambda_d n_d,$$

where $\lambda_i \geq 0$ and not all α_i are zero. Then

$$\langle b_{d+1}, n \rangle = \lambda_1 \langle b_{d+1}, n_1 \rangle + \lambda_2 \langle b_{d+1}, n_2 \rangle + \dots + \lambda_d \langle b_{d+1}, n_d \rangle < 0,$$

a contradiction. □

Lemma 4.2. *Let $\varepsilon \leq \frac{1}{3(d+1)^3}$, $a = \frac{1}{d+1} + \varepsilon$. Assume that*

$$H_1, H_2, \dots, H_{d+1}$$

be a generating $(d + 1)$ -tuple of half-spaces of weight a and

$$B_1, B_2, \dots, B_{d+1}$$

the corresponding $(d + 1)$ -tuple of simplicial cones. Then

1. $\sum_{i=1}^{d+1} \mu(B_i) \geq 1 - (d+1)\varepsilon.$
2. $\frac{1}{d+1} - (2d+1)\varepsilon \leq \mu(B_i) \leq \frac{1}{d+1} + \varepsilon.$

Proof. One can see that the set $\bigcup_{i=1}^{d+1} B_i$ is exactly the region that is covered exactly d times by the covering family of half-spaces

$$H_1, H_2, \dots, H_{d+1}.$$

All other points of \mathbb{R}^d points are covered at most $d-1$ times. Hence

$$d \sum_{i=1}^{d+1} \mu(B_i) + (d-1) \left(1 - \sum_{i=1}^{d+1} \mu(B_i) \right) \geq (d+1)(1-a),$$

or

$$(d-1) + \sum_{i=1}^{d+1} \mu(B_i) \geq d - (d+1)\varepsilon.$$

This proves assertion 1.

Without loss of generality let

$$\mu(B_1) \leq \mu(B_2) \leq \dots \leq \mu(B_{d+1}).$$

Assume that $\mu(B_{d+1}) > \frac{1}{d+1} + \varepsilon$. Then

$$\mu(H_{d+1}) \leq 1 - \mu(B_{d+1}) < \frac{d}{d+1} - \varepsilon = 1 - a.$$

Hence the $(d+1)$ -tuple of H_i cannot have weight a , a contradiction.

Assume that $\mu(B_1) < \frac{1}{d+1} - (2d+1)\varepsilon$. Then

$$\sum_{i=2}^{d+1} \mu(B_i) \geq \frac{d}{d+1} + d\varepsilon.$$

Hence $\mu(B_{d+1}) > \frac{1}{d+1} + \varepsilon$, which is not possible, as shown above. Thus

$$\frac{1}{d+1} - (2d+1)\varepsilon \leq \mu(B_1) \leq \mu(B_2) \leq \dots \leq \mu(B_{d+1}) \leq \frac{1}{d+1} + \varepsilon,$$

and assertion 2 is proved. □

Lemma 4.3 (Main Lemma). *Let $\varepsilon \leq \frac{1}{3(d+1)^3}$, $a = \frac{1}{d+1} + \varepsilon$. Assume that*

$$H_1, H_2, \dots, H_{d+1} \quad \text{and} \quad H'_1, H'_2, \dots, H'_{d+1}$$

are two generating $(d+1)$ -tuples of half-spaces with weight at most a . Let

$$B_1, B_2, \dots, B_{d+1} \quad \text{and} \quad B'_1, B'_2, \dots, B'_{d+1}$$

be the respective $(d+1)$ -tuples of simplicial cones. Then there is a permutation σ of the set $\{1, 2, \dots, d+1\}$ such that for every $i, j \in \{1, 2, \dots, d+1\}$, $i \neq j$

$$\mu(B_i \cap B'_{\sigma(i)}) \geq \frac{1}{d+1} - (3d+2)\varepsilon \quad \mu(B_i \cap B'_{\sigma(j)}) = 0.$$

Proof. Consider the bipartite graph G with $V(G) = V_1 \cup V_2$, where

$$V_1 = \{B_1, B_2, \dots, B_{d+1}\}, \quad V_2 = \{B'_1, B'_2, \dots, B'_{d+1}\}.$$

Let

$$(B_i, B'_j) \in E(G) \iff \mu(B_i \cap B'_j) > 0.$$

We claim that G is a perfect matching. First we prove that G has a perfect matching as a subgraph.

Assume that there is no perfect matching in G . According to Hall's marriage lemma, we may assume, up to a permutation of indices, that B_1, B_2, \dots, B_m are connected only with B'_1, B'_2, \dots, B'_k , and $k < m$.

Thus the set $\bigcup_{i=1}^m B_i$ is covered by the set

$$\bigcup_{i=1}^k B'_i \cup \left(\mathbb{R}^d \setminus \bigcup_{i=1}^{d+1} B'_i \right).$$

Hence

$$\bigcup_{i=m+1}^{d+1} B_i \cup \left(\mathbb{R}^d \setminus \bigcup_{i=1}^{d+1} B_i \right) \cup \bigcup_{i=1}^k B'_i \cup \left(\mathbb{R}^d \setminus \bigcup_{i=1}^{d+1} B'_i \right) = \mathbb{R}^d,$$

and, consequently,

$$\sum_{i=m+1}^{d+1} \mu(B_i) + \mu\left(\mathbb{R}^d \setminus \bigcup_{i=1}^{d+1} B_i\right) + \sum_{i=1}^k \mu(B'_i) + \mu\left(\mathbb{R}^d \setminus \bigcup_{i=1}^{d+1} B'_i\right) \geq 1.$$

Using Lemma 4.2, we get

$$(d+1+k-m)\left(\frac{1}{d+1} + \varepsilon\right) + 2(d+1)\varepsilon \geq 1.$$

Since

$$\varepsilon \leq \frac{1}{3(d+1)^3} < \frac{1}{(d+1)(3d+2)},$$

this is a contradiction. Hence G contains a perfect matching.

Up to a permutation of indices we can assume that $(B_i, B'_i) \in E(G)$ for each i . Next we show that no other edge of G exists.

Without loss of generality assume that $(B_1, B'_2) \in E(G)$. Choose a point $b_1 \in \text{int } B_1 \cap B'_2$. For each $i > 1$ choose a point $b_i \in B_i \cap B'_i$.

We have $b_i \in \int B_i$ for each i . Hence, by Lemma 4.1,

$$\mathbf{0} \in \text{int conv}\{b_1, b_2, \dots, b_{d+1}\}.$$

On the other hand,

$$\mathbf{0} \notin \text{int } H'_1 \supset \text{conv}\{b_1, b_2, \dots, b_{d+1}\},$$

a contradiction. Therefore G is exactly a perfect matching.

Finally, according to Lemma 4.2, $\mu(B_i) \geq \frac{1}{d+1} - (2d+1)\varepsilon$, and

$$\mu(B_i \setminus B'_i) \leq \mu\left(\mathbb{R}^d \setminus \bigcup_{i=1}^{d+1} B'_i\right) \leq (d+1)\varepsilon.$$

Hence

$$\mu(B_i \cap B'_i) \geq \frac{1}{d+1} - (3d+2)\varepsilon.$$

□

5 The central ray of a simplicial cone

In this section μ denotes a nice measure in \mathbb{R}^d .

Let B be a simplicial cone in \mathbb{R}^d with vertex $\mathbf{0}$. For $\alpha \in (0, 1]$ define

$$\mathcal{H}(B, \alpha) = \{H : H \text{ is a half-space, } \mathbf{0} \in \partial H, \text{ and } \mu(H \cap B) \geq \alpha \mu(B)\}.$$

Choose an arbitrary unit vector n such that $\langle n, b \rangle > 0$ for every $b \in B \setminus \mathbf{0}$. Since B contains no straight line, such n exists.

Consider the central projection π_c of \mathbb{R}^d with the center at $\mathbf{0}$ onto the plane

$$\Pi = \{y : \langle n, y \rangle = 1\}.$$

Define the probability measure μ_c in the plane Π as follows:

$$\mu_c(X) = \frac{\mu(\pi_c^{-1}(X) \cap B)}{\mu(B)}$$

for every measurable $X \subseteq \Pi$.

Let H be a half-space in \mathbb{R}^d such that ∂H is not orthogonal to n . Then $H \cap \Pi$ is a half-space in Π and

$$\mu_c(H \cap \Pi) = \mu(H \cap B).$$

If $\alpha \geq \frac{d-1}{d}$, then, by Radó's theorem,

$$\bigcap_{H \in \mathcal{H}(B, \alpha)} (H \cap \Pi) \neq \emptyset.$$

The intersection above is contained in B , hence there exists a non-zero vector $c \in B$ such that

$$\bigcap_{H \in \mathcal{H}(B, \alpha)} (H \cap B) \supseteq \{\lambda c : \lambda \geq 0\}.$$

The above argument for $\alpha = \frac{d}{d+1} > \frac{d-1}{d}$ implies that the set

$$C(B) = \bigcap_{H \in \mathcal{H}\left(B, \frac{d}{d+1}\right)} (H \cap B)$$

is a convex cone of positive measure. We will call $C(B)$ the *central cone* of B .

Let \mathbb{S}^{d-1} be the unit sphere in \mathbb{R}^d . Define

$$e(B) = \frac{\int_{\mathbb{S}^{d-1} \cap C(B)} x \, dx}{\left\| \int_{\mathbb{S}^{d-1} \cap C(B)} x \, dx \right\|},$$

where dx is the standard volume form in \mathbb{S}^{d-1} . We will call $e(B)$ the *central vector* of B , and the ray along $e(B)$ the *central ray* of B .

Emphasize that $e(B)$ is a unit vector, and that $e(B) \in C(B)$.

For the next proposition we do not give an explicit proof, but rather notice that the proof does not require any non-standard arguments.

Proposition 5.1. *$C(B)$ and $e(B)$ change continuously with a continuous change of μ and B .*

Remark. Actually, we need to define a continuous change of a cone. To do this, it is enough to define a basic neighborhood of a cone. Given $C(B)$, choose an arbitrary pair of closed convex cones C_{int} and C_{ext} such that

$$C_{int} \subset \text{int} C(B), \quad C(B) \subset \text{int} C_{ext}.$$

Then C_{int} and C_{ext} span the following basic neighborhood of $C(B)$: the set of all closed cones C satisfying

$$C_{int} \subset \text{int} C, \quad C \subset \text{int} C_{ext}.$$

Basic neighborhoods of B are defined in a similar way.

We will also need the following Lemma 5.2 and Corollary 5.3.

Lemma 5.2. *Let B and B' be simplicial cones, both with vertex $\mathbf{0}$. Suppose that*

$$\frac{1}{d+1} - \frac{2d+1}{3(d+1)^3} \leq \mu(B), \mu(B') \leq \frac{1}{d+1} + \frac{1}{3(d+1)^3},$$

$$\mu(B \cap B') \geq \frac{1}{d+1} - \frac{3d+2}{3(d+1)^3}.$$

Then

$$B \supseteq C(B') \quad \text{and} \quad B' \supseteq C(B).$$

Proof. The conditions are symmetric for B and B' , so it is enough to prove that $B \supseteq C(B')$. Let $C(B') \setminus B \neq \emptyset$. Then there exists a half-space H such that

$$\mathbf{0} \in \partial H, \quad B \subset H, \quad \text{and} \quad C(B') \setminus H \neq \emptyset.$$

But we have

$$\mu(H \cap B') \geq \mu(B \cap B') \geq \frac{1}{d+1} - \frac{3d+2}{3(d+1)^3} \geq \frac{d}{d+1} \left(\frac{1}{d+1} + \frac{1}{3(d+1)^3} \right) \geq \frac{d}{d+1} \mu(B').$$

Hence $C(B') \subset H$ by definition of $C(B')$. A contradiction. □

Corollary 5.3. *If B, B' are as in Lemma 5.2, then*

$$e(B) \in B' \quad \text{and} \quad e(B') \in B.$$

6 Ordered $(d+1)$ -tuples of small weight

For brevity, we will write

$$a_0 = \frac{1}{d+1} + \frac{1}{3(d+1)^3}.$$

Hence our main result, Theorem 1.2 says that for a measure μ in \mathbb{R}^{d+1} there exists a line with depth a_0 or greater. We remind that μ is considered to be nice (see the definition in Section 2) because of a standard approximation argument.

In this section we assume that the statement of Theorem 1.2 is false. That is, for every direction $\ell \in \mathbb{R}P^d$ of a line in \mathbb{R}^{d+1} the projected measure μ_ℓ is such that no point in \mathbb{R}^d has depth a_0 or greater.

Recall that we consider the quotient tautological bundle $\xi = (E, \mathbb{R}P^d, p)$. Each fiber $p^{-1}(\ell)$ is isomorphic to \mathbb{R}^d and can be treated as the domain for the

projected measure μ_ℓ . Due to the uniqueness and continuity of the centerpoint under our assumptions (Lemma 3.2), we can set the origin of the fiber $p^{-1}(\ell)$ to be the centerpoint of μ_ℓ .

Define

$$a_1 = \frac{1}{2} \left(a_0 + \max_{\ell \in \mathbb{R}P^d} \text{depth}_{\mu_\ell}(\mathbf{0}) \right).$$

(Due to continuity of $\text{depth}_{\mu_\ell}(\mathbf{0})$, the maximum exists.) Since

$$\max_{\ell \in \mathbb{R}P^d} \text{depth}_{\mu_\ell}(\mathbf{0}) < a_0,$$

we have

$$\text{depth}_{\mu_\ell}(\mathbf{0}) < a_1 < a_0$$

for each $\ell \in \mathbb{R}P^d$.

Given $\ell \in \mathbb{R}P^d$ and $a \in (0, 1)$, consider the fiber $p^{-1}(\ell)$ with the measure μ_ℓ . For these fiber and measure, denote by $\mathcal{R}_\ell(a)$ the family of all (unordered) generating $(d+1)$ -tuples of half-spaces with weight at most a .

Rather than the family $\mathcal{R}_\ell(a)$ of unordered $(d+1)$ -tuples we want to have a family $\mathcal{R}_\ell^*(a)$ of ordered $(d+1)$ -tuples with the following natural properties.

- (R1) There is a bijection between $\mathcal{R}_\ell(a)$ and $\mathcal{R}_\ell^*(a)$: every unordered $(d+1)$ -tuple from $\mathcal{R}_\ell(a)$ maps to an ordered $(d+1)$ -tuple from $\mathcal{R}_\ell^*(a)$ consisting of the same half-spaces.
- (R2) If $(H_1, H_2, \dots, H_{d+1})$ and $(H'_1, H'_2, \dots, H'_{d+1})$ are two elements of $\mathcal{R}_\ell^*(a)$, and $(B_1, B_2, \dots, B_{d+1})$ and $(B'_1, B'_2, \dots, B'_{d+1})$ are the corresponding $(d+1)$ -tuples of cones, then $\mu_\ell(B_i \cap B'_i) \geq \frac{1}{d+1} - \frac{3d+2}{3(d+1)^3}$.

Lemma 6.1. *Let a satisfy*

$$\text{depth}_{\mu_\ell}(\mathbf{0}) < a \leq \frac{1}{d+1} + \frac{1}{3(d+1)^3}.$$

Then there exists a family of ordered $(d+1)$ -tuples of half-spaces $\mathcal{R}_\ell^(a)$ satisfying the conditions (R1) and (R2).*

Proof. Choose an arbitrary unordered $(d+1)$ -tuple from $\mathcal{R}_\ell(a)$ and select an arbitrary order for it, say,

$$(H_1, H_2, \dots, H_{d+1}).$$

According to Lemma 4.3, for any unordered $(d+1)$ -tuple from $\mathcal{R}_\ell(a)$ one can choose a unique order

$$(H'_1, H'_2, \dots, H'_{d+1})$$

such that

$$\mu_\ell(B_i \cap B'_i) \geq \frac{1}{d+1} - \frac{3d+2}{3(d+1)^3} \quad \mu_\ell(B_i \cap B'_j) = 0,$$

where B_i and B'_i are the respective simplicial cones. Let $\mathcal{R}_\ell^*(a)$ be the family of all such ordered $(d+1)$ -tuples.

Let

$$(H'_1, H'_2, \dots, H'_{d+1}), (H''_1, H''_2, \dots, H''_{d+1}) \in \mathcal{R}_\ell^*(a).$$

Then

$$\mu_\ell(B_i \cap B'_i) > \frac{1}{2}\mu_\ell(B_i), \quad \mu_\ell(B_i \cap B''_i) > \frac{1}{2}\mu_\ell(B_i).$$

Therefore

$$\mu_\ell(B'_i \cap B''_i) > 0.$$

Lemma 4.3 implies

$$\mu_\ell(B'_i \cap B''_i) \geq \frac{1}{d+1} - \frac{3d+2}{3(d+1)^3}.$$

Hence $\mathcal{R}_\ell^*(a)$ meets the required conditions. □

One can notice that the proof above leads to the following.

Proposition 6.2. *If a family $\mathcal{R}_\ell^*(a)$ satisfies the conditions (R1) and (R2), then any other family satisfying these conditions is obtained by choosing an arbitrary permutation σ and applying it to each element of $\mathcal{R}_\ell^*(a)$.*

In the hypothesis of the next Lemma 6.3 we are given a direction $\ell \in \mathbb{R}P^d$, its neighborhood $U_0 \subset \mathbb{R}P^d$, and an isomorphism $p^{-1}(U_0) \cong U_0 \times \mathbb{R}^d$. The isomorphism is required to match each $p^{-1}(\ell')$ to $\{\ell'\} \times \mathbb{R}^d$, and the set of origins of fibers to $U_0 \times \{\mathbf{0}\}$. (Notice that such an isomorphism always exists if U_0 is small enough.)

Let $t : U_0 \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be the projection that erases the U_0 -coordinate. Then the family $\mathcal{R}_{\ell'}^*(a)$, which is defined in $p^{-1}(\ell')$ ($\cong \ell' \times \mathbb{R}^d$) has a natural counterpart $t(\mathcal{R}_{\ell'}^*(a))$ in \mathbb{R}^d , the image space of t . A measure $\mu_{\ell'}$ in $p^{-1}(\ell')$ has also a counterpart $\mu_{\ell'}^t$ in \mathbb{R}^d .

Lemma 6.3. *Let a direction $\ell \in \mathbb{R}P^d$, its neighborhood $U_0 \subset \mathbb{R}P^d$, and an isomorphism $p^{-1}(U_0) \cong U_0 \times \mathbb{R}^d$ be as above. Then there is a neighborhood $U \subset U_0$ of ℓ and a choice of order for each $\mathcal{R}_{\ell'}^*(a_1)$ (ℓ' runs through U) such that*

$$t(\mathcal{R}_{\ell'}^*(a_1)) \subset t(\mathcal{R}_\ell^*(a_0)).$$

Proof. Let, $f_{\ell'} : \mathbb{R}^d \rightarrow \mathbb{R}_+$ denote the density function of the measure $\mu_{\ell'}^t$. Choose $U \subseteq U_0$ small enough to satisfy

$$\|f_{\ell'} - f_{\ell}\|_{L^1} < a_0 - a_1 \quad \text{for every } \ell' \in U.$$

If $\ell' \in U$ and a half-space $H \subset \mathbb{R}^d$ satisfies $\mu_{\ell'}^t(H) \leq a_1$, then $\mu_{\ell}^t(H) < a_0$.
Therefore we have

$$(H_1, H_2, \dots, H_{d+1}) \in \mathcal{R}_{\ell}(a_0)$$

whenever

$$(H_1, H_2, \dots, H_{d+1}) \in \mathcal{R}_{\ell'}(a_1).$$

For each $(d+1)$ -tuple from $\mathcal{R}_{\ell'}(a_1)$ choose the same order as in $\mathcal{R}_{\ell}^*(a_0)$. The resulting family of ordered $(d+1)$ -tuples satisfies the conditions (R1) and (R2) above and therefore is suitable to be $\mathcal{R}_{\ell'}^*(a_1)$. □

Let us consider the set of all $(d+1)$ -tuples of half-spaces in \mathbb{R}^d with $\mathbf{0}$ on the boundary. We can consider it as $(\mathbb{S}^{d-1})^{d+1}$ with the natural topology. We will need the following lemma, formalizing the fact that generating $(d+1)$ -tuples of small weight are far enough from non-generating $(d+1)$ -tuples.

Lemma 6.4. *For every $\ell \in \mathbb{R}P^d$, $0 < a \leq a_0$ the set $\mathcal{R}_{\ell}^*(a)$ is compact.*

Proof. Throughout the proof the notation relates to the space $p^{-1}(\ell)$ and the measure μ_{ℓ} .

Let the sequence

$$(H_1^{(j)}, H_2^{(j)}, \dots, H_{d+1}^{(j)}) \in \mathcal{R}_{\ell}^*(a)$$

$(j = 1, 2, \dots)$ converge and

$$H_i = \lim_{j \rightarrow \infty} H_i^{(j)}.$$

Let

$$(B_1^{(j)}, B_2^{(j)}, \dots, B_{d+1}^{(j)})$$

be the respective $(d+1)$ -tuples of simplicial cones.

By property (R2) of $\mathcal{R}_{\ell}^*(a)$, we have

$$\mu_{\ell}(B_i^{(1)} \cap B_i^{(j)}) \geq \frac{1}{d+1} - \frac{3d+2}{3(d+1)^3}.$$

Lemma 5.2 implies

$$C(B_i^{(1)}) \subseteq B_i^{(j)}.$$

Therefore, for each $1 \leq i, i' \leq d+1$, $i \neq i'$ one has

$$C(B_i^{(1)}) \subset \mathbb{R}^d \setminus H_i^{(j)}, \quad C(B_{i'}^{(1)}) \subset H_{i'}^{(j)}.$$

For each $i = 1, 2, \dots, d+1$ take a unit vector b_i pointing to the interior of $C(B_i^{(1)})$. If H is a half-space with $\mathbf{0} \in \partial H$, let $n(H)$ denote the outer unit normal to ∂H .

As b_i is separated from the boundary of $C(B_i)$, there exists $\delta > 0$, independent of j , such that for any $1 \leq i, i' \leq d+1$, $i \neq i'$ one has

$$\langle n(H_i^{(j)}), b_i \rangle \geq \delta, \quad \langle n(H_{i'}^{(j)}), b_i \rangle \leq -\delta.$$

Taking the limit, we obtain

$$\langle n(H_i), b_i \rangle \geq \delta, \quad \langle n(H_{i'}), b_i \rangle \leq -\delta.$$

By Lemma 4.1,

$$\mathbf{0} \in \text{int conv}\{b_1, b_2, \dots, b_{d+1}\}.$$

Hence, if H_i^* is a half-space such that $b_i = n(H_i^*)$, then the $(d+1)$ -tuple

$$(H_1^*, H_2^*, \dots, H_{d+1}^*)$$

is generating. Let

$$(B_1^*, B_2^*, \dots, B_{d+1}^*)$$

be the corresponding $(d+1)$ -tuple of simplicial cones. Then one has

$$n(H_i) \in \text{int } B_i^*.$$

Lemma 4.1 immediately yields

$$\mathbf{0} \in \text{int conv}\{n(H_1), n(H_2), \dots, n(H_{d+1})\}.$$

Hence the $(d+1)$ -tuple

$$(H_1, H_2, \dots, H_{d+1})$$

is generating and has the corresponding $(d+1)$ -tuple

$$(B_1, B_2, \dots, B_{d+1})$$

of simplicial cones.

By continuity of the weight function, the $(d+1)$ -tuple

$$(H_1, H_2, \dots, H_{d+1})$$

has weight at most a .

Finally, suppose that

$$(H_1, H_2, \dots, H_{d+1}) \notin \mathcal{R}_\ell^*(a).$$

Then there is a non-trivial permutation σ such that

$$(H_{\sigma(1)}, H_{\sigma(2)}, \dots, H_{\sigma(d+1)}) \in \mathcal{R}_\ell^*(a).$$

Due to the property (R2), for each $j = 1, 2, \dots$ we have

$$\mu_\ell(B_{\sigma(i)} \cap B_i^{(j)}) \geq \frac{1}{d+1} - \frac{3d+2}{3(d+1)^3}.$$

Taking the limit yields

$$\mu_\ell(B_{\sigma(i)} \cap B_i) \geq \frac{1}{d+1} - \frac{3d+2}{3(d+1)^3},$$

which is impossible. A contradiction shows that

$$(H_1, H_2, \dots, H_{d+1}) \in \mathcal{R}_\ell^*(a).$$

Thus $\mathcal{R}_\ell^*(a)$ is closed and hence compact.

□

7 Constructing the hypothetical $(d+1)$ -fold space cover

$$E' \subset E(\xi)$$

We continue the argument using the assumptions and the notation of the previous section.

Let $\ell \in \mathbb{R}P^d$ and

$$(n_1, n_2, \dots, n_{d+1})$$

be an arbitrary $(d+1)$ -tuple of unit vectors in $p^{-1}(\ell)$. (Note that the space of all such $(d+1)$ -tuples is $(\mathbb{S}^{d-1})^{d+1}$.) If H_i is a half-space of $p^{-1}(\ell)$ such that $\mathbf{0} \in \partial H_i$ and n_i is the outer normal to ∂H_i , and the $(d+1)$ -tuple

$$(H_1, H_2, \dots, H_{d+1})$$

is generating, then we write $\text{weight}_\ell(n_1, n_2, \dots, n_{d+1})$ for the weight of that $(d+1)$ -tuple of half-spaces.

Let $\mathcal{R}_\ell^*(a_1)$ satisfy the properties (R1) and (R2) from the previous section. (That means, we have $(d+1)!$ candidates for $\mathcal{R}_\ell^*(a_1)$ according to Lemma 6.1, but we choose one of them, and fix the choice for the next definition.)

Now we define the vector function

$$e_i(\ell, n_1, n_2, \dots, n_{d+1}) : \mathbb{R}P^d \times (\mathbb{S}^{d-1})^{d+1} \rightarrow p^{-1}(\ell).$$

The definition will consist of two mutually disjoint cases.

Case 1. $(H_1, H_2, \dots, H_{d+1}) \notin \mathcal{R}_\ell^*(a_1)$. Put

$$e_i(\ell, n_1, n_2, \dots, n_{d+1}) = \mathbf{0}.$$

Case 2. $(H_1, H_2, \dots, H_{d+1}) \in \mathcal{R}_\ell^*(a_1)$. Put

$$e_i(\ell, n_1, n_2, \dots, n_{d+1}) = (a_1 - \text{weight}_\ell(n_1, n_2, \dots, n_{d+1}))e(B_i),$$

where $(B_1, B_2, \dots, B_{d+1})$ is the $(d+1)$ -tuple of simplicial cones corresponding to the generating $(d+1)$ -tuple $(H_1, H_2, \dots, H_{d+1})$.

Now we are ready to define the $(d+1)$ -tuple of vectors $E' \cap p^{-1}(\ell)$. Namely, the i -th element of the $(d+1)$ -tuple is defined as follows:

$$e_i(\ell) = \int_{(\mathbb{S}^{d-1})^{d+1}} e_i(\ell, n_1, n_2, \dots, n_{d+1}) dn_1 dn_2 \dots dn_{d+1}. \quad (7)$$

We emphasize that the ordered $(d+1)$ -tuple

$$(e_1(\ell), e_2(\ell), \dots, e_{d+1}(\ell))$$

depends on the choice of $\mathcal{R}_\ell^*(a_1)$ from $(d+1)!$ possible variants, but the unordered $(d+1)$ -tuple

$$\{e_1(\ell), e_2(\ell), \dots, e_{d+1}(\ell)\}$$

does not.

In order to continue, we first clarify the notation. Consider an arbitrary direction $\ell_0 \in \mathbb{R}P^d$. Then there is a small neighborhood U_0 of ℓ_0 such that there exists an isomorphism between $p^{-1}(U_0)$ and $U_0 \times \mathbb{R}^d$. We will consider the projection

$$t : p^{-1}(U_0) \cong U_0 \times \mathbb{R}^d \rightarrow \mathbb{R}^d$$

that drops the U_0 -component of the Cartesian product. Further, reduce U_0 to a smaller neighborhood $U_1 \subseteq U_0$ of ℓ_0 so that one can choose $\mathcal{R}_\ell^*(a_1)$ for each $\ell \in U_1$ consistently in the sense of Lemma 6.3.

There is a map

$$\hat{e}_i : U_1 \times (\mathbb{S}^{d-1})^{d+1} \rightarrow \mathbb{R}^d$$

acting as follows. The input is $\ell \in U_1$ and $d + 1$ unit vectors

$$\hat{n}_1, \hat{n}_2, \dots, \hat{n}_{d+1} \in \mathbb{R}^d.$$

If $n_j \in p^{-1}(\ell)$ is such that $t(n_j) = \hat{n}_j$, then

$$\hat{e}_i = t \circ e_i(\ell, n_1, n_2, \dots, n_{d+1}).$$

Lemma 7.1. *The map \hat{e}_i is continuous.*

Proof. As this will cause no ambiguity, write e_i instead of \hat{e}_i , μ_ℓ instead of $t(\mu_\ell)$ and $\mathcal{R}_\ell^*(a)$ instead of $t(\mathcal{R}_\ell^*(a))$. In other words, we treat the measures μ_ℓ and the families $\mathcal{R}_\ell^*(a)$ as objects placed in the same \mathbb{R}^d .

Consider the two cases.

Case 1. $e_i(\ell, n_1, n_2, \dots, n_{d+1}) = \mathbf{0}$. If, as before, H_j is a half-space with outer normal n_j and $\mathbf{0} \in \partial H_j$, then there are four subcases.

Subcase 1.1. $(H_1, H_2, \dots, H_{d+1})$ is not a generating $(d+1)$ -tuple. Then Lemma 6.4 immediately implies that any $(d+1)$ -tuple close enough to $(H_1, H_2, \dots, H_{d+1})$ cannot belong to $\mathcal{R}_{\ell_0}^*(a_0)$, and hence to $\mathcal{R}_{\ell'}^*(a_1)$ for any $\ell' \in U_1$. Thus

$$e_i(\ell', n'_1, n'_2, \dots, n'_{d+1}) \equiv \mathbf{0}$$

for any $\ell' \in U_1$ and any $(n'_1, n'_2, \dots, n'_{d+1})$ close enough to $(n_1, n_2, \dots, n_{d+1})$.

Subcase 1.2. $\text{weight}_\ell(n_1, n_2, \dots, n_{d+1}) > a_1$. Then for any $(\ell', n'_1, n'_2, \dots, n'_{d+1})$ close enough to $(\ell, n_1, n_2, \dots, n_{d+1})$ we have

$$\text{weight}_{\ell'}(n'_1, n'_2, \dots, n'_{d+1}) > a_1,$$

and, consequently,

$$e_i(\ell', n'_1, n'_2, \dots, n'_{d+1}) \equiv \mathbf{0}.$$

Subcase 1.3. There exists a non-trivial permutation σ such that

$$(H_{\sigma(1)}, H_{\sigma(2)}, \dots, H_{\sigma(d+1)}) \in \mathcal{R}_\ell^*(a_1).$$

Then

$$(H_{\sigma(1)}, H_{\sigma(2)}, \dots, H_{\sigma(d+1)}) \in \mathcal{R}_{\ell_0}^*(a_0),$$

and, therefore,

$$(H_1, H_2, \dots, H_{d+1}) \notin \mathcal{R}_{\ell_0}^*(a_0).$$

The rest of the argument goes as in Subcase 1.1.

Subcase 1.4. $(H_1, H_2, \dots, H_{d+1}) \in \mathcal{R}_\ell^*(a_1)$, $\text{weight}_\ell(n_1, n_2, \dots, n_{d+1}) = a_1$. Then, of course, the $(d+1)$ -tuple $(H_1, H_2, \dots, H_{d+1})$ is generating, and so are all close

enough $(d + 1)$ -tuples of half-spaces. Moreover, for any $\delta > 0$ we can restrict ourselves to $(\ell', n'_1, n'_2, \dots, n'_{d+1})$ close enough to $(\ell, n_1, n_2, \dots, n_{d+1})$ so that

$$\text{weight}_{\ell'}(n'_1, n'_2, \dots, n'_{d+1}) > a_1 - \delta.$$

The last inequality implies

$$\|e_i(\ell', n'_1, n'_2, \dots, n'_{d+1})\| < \delta,$$

which is sufficient since δ is arbitrary.

Case 2. $e_j(\ell', n'_1, n'_2, \dots, n'_{d+1}) \neq \mathbf{0}$.

Let, as above, $(B_1, B_2, \dots, B_{d+1})$ be the $(d + 1)$ -tuple of simplicial cones corresponding to the generating $(d + 1)$ -tuple $(H_1, H_2, \dots, H_{d+1})$, and $C(B_j)$ — the central cone of B_j with respect to the measure μ_ℓ .

Given $\delta > 0$, choose two closed convex cones C_{int} and C_{ext} such that

1. $C_{int} \subset \text{int} C(B_i), C(B_j) \subset \text{int} C_{ext}$
2. For every convex cone C such that $C_{int} \subseteq C \subseteq C_{ext}$ one has

$$\|e - e(B_i)\| \leq \delta,$$

where

$$e = \frac{\int_{\mathbb{S}^{d-1} \cap C} x dx}{\left\| \int_{\mathbb{S}^{d-1} \cap C} x dx \right\|}.$$

(dx is the usual volume form in \mathbb{S}^{d-1} .)

For $(\ell', n'_1, n'_2, \dots, n'_{d+1})$ close enough to $(\ell, n_1, n_2, \dots, n_{d+1})$ we can satisfy the following three conditions. (Here H'_i denotes the half-space with the outer normal n'_i and $\mathbf{0} \in \partial H'_i$.)

1. $(H'_1, H'_2, \dots, H'_{d+1}) \in R_{\ell'}^*(a_1)$. Hence there is a $(d + 1)$ -tuple of simplicial cones $(B'_1, B'_2, \dots, B'_{d+1})$ corresponding to the generating $(d + 1)$ -tuple $(H'_1, H'_2, \dots, H'_{d+1})$.
2. $C_{int} \subset C(B'_i) \subset C_{ext}$, where $C(B'_i)$ is the central cone of B'_i with respect to the measure $\mu_{\ell'}$.
3. $|\text{weight}_{\ell'}(n'_1, n'_2, \dots, n'_{d+1}) - \text{weight}_\ell(n_1, n_2, \dots, n_{d+1})| < \delta$.

We will write w and w' for $\text{weight}_\ell(n_1, n_2, \dots, n_{d+1})$ and $\text{weight}_{\ell'}(n'_1, n'_2, \dots, n'_{d+1})$, respectively.

Under all the conditions above, we have

$$\begin{aligned} \|e_i(\ell', n'_1, n'_2, \dots, n'_{d+1}) - e_i(\ell, n_1, n_2, \dots, n_{d+1})\| &= \|w \cdot e(B_i) - w' \cdot e(B'_i)\| \leq \\ &w \|e(B_i) - e(B'_i)\| + |w - w'| \|e(B'_i)\| \leq 2\delta. \end{aligned}$$

As a consequence, given $\delta > 0$, we have found a neighborhood of $(\ell', n_1, n_2, \dots, n_{d+1})$ such that e_j changes at most by 2δ . □

Lemma 7.2. *The (unordered) $(d+1)$ -tuple*

$$\{e_1(\ell), e_2(\ell), \dots, e_{d+1}(\ell)\}$$

depends continuously on ℓ .

Proof. Since Cauchy and Heine definitions of continuity are equivalent in our case, we will use the latter.

Consider an arbitrary sequence $\{\ell_j\} \rightarrow \ell$. By Lemma 7.1, the sub-integral function for ℓ_j in (7) pointwise converges to that of ℓ . Hence $e_i(\ell_j)$ converges to $e_i(\ell)$ by the Bounded Convergence Theorem (see, for example, [7, Section 4.2]). □

Lemma 7.3. *If $\ell \in \mathbb{R}P^d$, $\mathbb{R}^d = p^{-1}(\ell)$, and $\mathbf{0}$ is the origin of \mathbb{R}^d , then*

$$\mathbf{0} \in \text{conv}\{e_1(\ell), e_2(\ell), \dots, e_{d+1}(\ell)\}. \quad (8)$$

Proof. Let $(H_1, H_2, \dots, H_{d+1}) \in \mathcal{R}_\ell^*(a_1)$. Let B_1, B_2, \dots, B_{d+1} be the corresponding $(d+1)$ -tuple of simplicial cones.

Choose an arbitrary $(d+1)$ -tuple

$$(H'_1, H'_2, \dots, H'_{d+1}) \in \mathcal{R}_\ell^*(a_1).$$

If $(B'_1, B'_2, \dots, B'_{d+1})$ is the corresponding $(d+1)$ -tuple of simplicial cones, and $n'_j = n(H'_j)$. Hence either

$$e_i(\ell, n'_1, n'_2, \dots, n'_{d+1}) = \mathbf{0} \quad (9)$$

(if $\text{weight}_\ell(n'_1, n'_2, \dots, n'_{d+1}) = a_1$), or

$$e_i(\ell, n'_1, n'_2, \dots, n'_{d+1}) \in \text{int } C(B'_i) \subset \text{int } B_i. \quad (10)$$

Therefore either (9) or (10) holds for all possible

$$(n'_1, n'_2, \dots, n'_{d+1}) \in (\mathbb{S}^{d-1})^{d+1},$$

and, due to choice of a_1 , (10) holds in a set of positive measure.

Consequently, (7) implies that

$$e_i(\ell) \in \text{int } B_i.$$

Applying Lemma 4.1, we immediately get (8). □

8 Final topological argument

Lemma 8.1. *Let $d \geq 2$, $\xi = (E, \mathbb{R}P^d, p)$ be the tautological quotient bundle. Then there is no $(d+1)$ -fold space cover $E' \subset E$ such that for every $\ell \in \mathbb{R}P^d$*

1. *The simplex $S(\ell) = E' \cap p^{-1}(\ell)$ is non-degenerate.*
2. *The origin of the fiber $p^{-1}(\ell)$ is an interior point of $S(\ell)$.*

Proof. Suppose that such E' exists. Since $\pi_1(\mathbb{R}P^d) = \mathbb{Z}_2$, then E' splits into 1-fold and 2-fold subcovers.

We will show then that in each case ξ admits a non-vanishing section. That is, there exists $F \subset E$ such that $p|_F$ is a homeomorphism from F to $\mathbb{R}P^d$, and for every $\ell \in \mathbb{R}P^d$ the point $F \cap p^{-1}(\ell)$ is not the origin of the fiber $p^{-1}(\ell)$.

If there is a 1-fold subcover $F \subset E$, then it is a non-vanishing section itself. If there is a 2-fold subcover $G \subset E'$, then for each $\ell \in \mathbb{R}P^d$ define

$$f(\ell) = g_1 + g_2, \quad \text{where } \{g_1, g_2\} = G \cap p^{-1}(\ell).$$

(The sum of vectors is well defined in the fiber $p^{-1}(\ell)$.) Put

$$F = \{f(\ell) : \ell \in \mathbb{R}P^d\}.$$

Since $d \geq 2$, and g_1 and g_2 are two vertices of the simplex $S(\ell)$, which contains the origin inside, the sum $g_1 + g_2$ cannot vanish. Hence F is a non-vanishing section.

Now notice that, if η is the tautological line bundle over $\mathbb{R}P^d$, then $\xi \oplus \eta$ is trivial. Hence for the corresponding Stiefel-Whitney classes we have

$$sw(\xi)sw(\eta) = 1$$

in $H^*(\mathbb{R}P^d, \mathbb{Z}_2) = \mathbb{Z}_2[x]/(x^{d+1})$, where $x \in H^1(\mathbb{R}P^d, \mathbb{Z}_2)$. But $sw(\eta) = 1 + x$, so $sw(\xi) = 1 + x + \dots + x^d$. (See the details, for example, in [4].)

Hence the top (d -th) Stiefel-Whitney class of ξ is non-zero. Therefore ξ cannot have a non-vanishing section, a contradiction. \square

Remark. The present version of Lemma 8.1 was suggested by R. Karasev and works for every dimension. For all dimensions, except $d = 3$ and $d = 7$ one can consider the tangent bundle of \mathbb{S}^d with an embedded $(d + 1)$ -fold cover. This gives $d + 1$ affinely independent vector fields in \mathbb{S}^d (and therefore d linearly independent vector fields), which is also impossible.

Now we are ready to finish the proof of our main result.

Proof of Theorem 1.2. We will write $d + 1$ rather than d for the dimension of the space. Assume that for some measure μ in \mathbb{R}^{d+1} no line has depth

$$a_0 = \frac{1}{d+1} + \frac{1}{3(d+1)^3}$$

or more.

Due to a standard approximation argument, we can assume that the measure μ in \mathbb{R}^{d+1} is nice (every measure can be approximated with nice measures).

For the quotient tautological bundle $\xi = (E, \mathbb{R}P^d, p)$ perform the construction from Section 7 to obtain E' — a $(d + 1)$ -fold space cover of $\mathbb{R}P^d$.

On the other hand, by Lemma 7.3 and Lemma 8.1 such a space cover cannot exist, a contradiction. \square

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